

AUTOMORPHISMS OF NON-SINGULAR NILPOTENT LIE ALGEBRAS

AROLDO KAPLAN AND ALEJANDRO TIRABOSCHI

ABSTRACT. For a real, non-singular, 2-step nilpotent Lie algebra \mathfrak{n} , the group $\text{Aut}(\mathfrak{n})/\text{Aut}_0(\mathfrak{n})$, where $\text{Aut}_0(\mathfrak{n})$ is the group of automorphisms which act trivially on the center, is the direct product of a compact group with the 1-dimensional group of dilations. Maximality of some automorphisms groups of \mathfrak{n} follows and is related to how close is \mathfrak{n} to being of Heisenberg type. For example, at least when the dimension of the center is two, $\dim \text{Aut}(\mathfrak{n})$ is maximal if and only if \mathfrak{n} is type H . The connection with fat distributions is discussed.

1. INTRODUCTION

A 2-step nilpotent real Lie algebra \mathfrak{n} with center \mathfrak{z} is called *non-singular* [E] if $\text{ad } x : \mathfrak{n} \rightarrow \mathfrak{z}$ is onto for any $x \notin \mathfrak{z}$. Equivalently, it is a vector-valued antisymmetric form

$$[\cdot, \cdot] : \mathfrak{v} \times \mathfrak{v} \rightarrow \mathfrak{z},$$

$\mathfrak{v} = \mathfrak{n}/\mathfrak{z}$, such that the 2-forms $\lambda([u, v])$ on \mathfrak{v} are non-degenerate for all $\lambda \in \mathfrak{z}^*$, $\lambda \neq 0$. We shall call such Lie algebras *fat algebras* for short, since they are the nilpotentizations, or symbols, of fat vector distributions. While for $m = 1$ there is only one fat algebra up to isomorphisms, for $m \geq 2$ there is an uncountable number of isomorphism classes and for $m \geq 3$ they form a wild set.

The group of automorphisms $\text{Aut}(\mathfrak{n})$ is the semidirect product of the group $G(\mathfrak{n})$ of graded automorphisms of $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with the abelian group $\text{Hom}(\mathfrak{v}, \mathfrak{z})$ times the group of dilations (t, t^2) . Hence, we concentrate on $G(\mathfrak{n})$. We prove that there is an exact sequence

$$1 \rightarrow G_0 \rightarrow G \rightarrow O(m)$$

where G_0 is the subgroup of G of elements that act trivially on the center and m is the dimension of this center. In other words, there are positive metrics (inner products) on \mathfrak{z} which are invariant under all of $\text{Aut}(\mathfrak{n})$. If a metric g is also given on \mathfrak{v} , as in the case of the nilpotentization of a subriemannian structure, we also consider the subgroups K_0, K , of graded automorphisms that leave g invariant, which define a compatible exact sequence

$$1 \rightarrow K_0 \rightarrow K \rightarrow O(m).$$

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Next, we compute the terms in this sequence and the images G/G_0 and K/K_0 , proving that the exactness of

$$1 \rightarrow \text{Lie}(K_0) \rightarrow \text{Lie}(K) \rightarrow \mathfrak{so}(m) \rightarrow 1$$

is equivalent to \mathfrak{n} being of Heisenberg type, while the exactness of

$$1 \rightarrow \text{Lie}(G_0) \rightarrow \text{Lie}(G) \rightarrow \mathfrak{so}(m) \rightarrow 1$$

is strictly more general. As to $G_0(\mathfrak{n})$, we describe it in detail for the case $m = 2$, leading a proof that, at least in that case, $\dim \text{Aut}(\mathfrak{n})$ is maximal if and only if \mathfrak{n} is of Heisenberg.

In the last section we explain the connection with the Equivalence Problem for fat subriemannian distributions.

Algebras of Heisenberg type, or *H-type*, arise as follows [K]. If \mathfrak{v} is a real unitary module over the Clifford algebra $\text{Cl}(\mathfrak{z})$ associated to a quadratic form on \mathfrak{z} , the identity

$$\langle z, [u, v] \rangle_{\mathfrak{z}} = \langle z \cdot u, v \rangle_{\mathfrak{v}}$$

with $z \in \mathfrak{z} \subset \text{Cl}(\mathfrak{z})$, $u, v \in \mathfrak{v}$, defines a fat $[\cdot, \cdot] : \mathfrak{v} \times \mathfrak{v} \rightarrow \mathfrak{z}$. Alternatively, they are characterized by possessing a positive-definite metric such that the operator $z \cdot$ defined by the above equation satisfies $z \cdot (z \cdot v) = -|z|^2 v$.

It follows from Adam's theorem on frames on spheres [H] that for any fat algebra there is an *H-type* algebra with the same $\dim \mathfrak{z}$ and $\dim \mathfrak{v}$. That these were, in some sense, the most symmetric, was expected from the properties of their sublaplacians [BTV] [CGN] [GV] [K], but we found no explicit statements in this regard. Finally, although the arguments below can be made more intrinsic, matrices are emphasized because they can be fed easily into MAGMA for the application of the methods of [DG].

2. AUTOMORPHISMS OF FAT ALGEBRAS

Let \mathfrak{n} be a 2-step Lie algebra with center \mathfrak{z} and let $\mathfrak{v} = \mathfrak{n}/\mathfrak{z}$, so that

$$(2.1) \quad \mathfrak{n} \cong \mathfrak{v} \oplus \mathfrak{z}$$

and the Lie algebra structure is encoded into the map

$$[\cdot, \cdot] : \Lambda^2 \mathfrak{v} \rightarrow \mathfrak{z}.$$

Let $n = \dim \mathfrak{v}$ and $m = \dim \mathfrak{z}$. Relative to a basis compatible with (2.1), the bracket becomes an \mathbb{R}^m -valued antisymmetric form on \mathbb{R}^n and an automorphism is a matrix of the form

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}, \quad a \in GL(n), b \in GL(m), c \in M_{n \times m}(\mathbb{R})$$

such that

$$b([u, v]) = [au, av].$$

$\text{Aut}(\mathfrak{n})$ always contains the normal subgroup $\mathfrak{D}(\mathfrak{n})$ of dilations and translations

$$\begin{pmatrix} tI_n & 0 \\ c & t^2 I_m \end{pmatrix}, \quad t \in \mathbb{R}^*, c \in M_{n \times m}(\mathbb{R}).$$

Let

$$G = G(\mathfrak{n}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \in SL(n), b \in GL(m), b([u, v]) = [au, av] \right\}.$$

Then $\text{Aut}(\mathfrak{n})$ is the semidirect product of $G(\mathfrak{n})$ with $\mathfrak{D}(\mathfrak{n})$. Let

$$G_0 = G_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & I_m \end{pmatrix}, a \in SL(n), [au, av] = [u, v] \right\},$$

the subgroup of automorphisms that act trivially on the center. These are Lie groups, G_0 is normal in G , and the quotient group

$$G/G_0$$

can be identified with the group of $b \in GL(\mathfrak{z})$ such that $b([u, v]) = [au, av]$ for some $a \in SL(\mathfrak{v})$. Obviously,

$$(2.2) \quad \dim \text{Aut}(\mathfrak{n}) = nm + 1 + \dim(G/G_0) + \dim(G_0).$$

Theorem 2.1. *Let \mathfrak{n} be a fat algebra with center \mathfrak{z} . Then there is a positive definite metric on \mathfrak{z} invariant under $G(\mathfrak{n})$.*

Proof. Fix arbitrary positive inner products on \mathfrak{v} and \mathfrak{z} . For $z \in \mathfrak{z}$, $u, v \in \mathfrak{v}$

$$(T_z u, v)_{\mathfrak{v}} = (z, [u, v])_{\mathfrak{z}}$$

defines a linear map $z \mapsto T_z$ from \mathfrak{z} to $\text{End}(\mathfrak{v})$. Clearly,

$$\mathfrak{n} \text{ fat} \Leftrightarrow T_z \in GL(\mathfrak{v}) \quad \forall z \neq 0.$$

Hence the hypothesis insures that the Pfaffian

$$P(z) = \det(T_z)$$

is non-zero on $\mathfrak{z} \setminus \{0\}$. This is a homogeneous polynomial of degree n , so it satisfies

$$(2.3) \quad k\|z\|^n \leq |P(z)| \leq K\|z\|^n$$

where k, K are the minimum and maximum values of $|P|$ on the unit sphere, which are positive.

Let now $g_{a,b} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \text{Aut}(\mathfrak{n})$. Then

$$T_{b^t z} = a^t T_z a$$

because $(T_{b^t z} u, v)_{\mathfrak{v}} = (b^t z, [u, v])_{\mathfrak{z}} = (z, b([u, v]))_{\mathfrak{z}} = (z, [au, av])_{\mathfrak{z}} = (T_z au, av)_{\mathfrak{v}} = (a^t T_z au, v)_{\mathfrak{v}}$. Consequently

$$P(b^t z) = (\det a)^2 P(z).$$

In particular, if $g \in G$ then $P(b^t z) = P(z)$. This implies

$$k\|b^t z\|^n \leq |P(b^t z)| = |P(z)| \leq K\|z\|^n$$

for all z , therefore $\|b\| \leq \sqrt[n]{K/k}$. The group of $b \in GL(\mathfrak{z})$ such that $g_{a,b} \in \text{Aut}(\mathfrak{n})$ for some $a \in SL(\mathfrak{v})$, is therefore bounded in $\text{End}(\mathbb{R}^m)$. Its closure is a compact Lie subgroup of $GL(\mathfrak{z})$, necessarily contained in $O(\mathfrak{z})$ for some positive definite metric. \square

From now on \mathfrak{z} will be assumed endowed with such invariant metric. If a metric g on \mathfrak{v} is also fixed, as in the case of the nilpotentization of a subriemannian structure, define the groups

$$K = K(\mathfrak{n}, g) = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, a \in SO(\mathfrak{v}), b \in O(\mathfrak{z}), [au, av] = b[u, v] \right\}$$

$$K_0 = K_0(\mathfrak{n}, g) = \left\{ \begin{pmatrix} a & 0 \\ 0 & I \end{pmatrix}, a \in SO(\mathfrak{v}), [au, av] = [u, v] \right\}.$$

Let $\mathfrak{g}, \mathfrak{g}_0, \mathfrak{k}, \mathfrak{k}_0$ be the Lie algebras of G, G_0, K, K_0 respectively. Then there is the commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathfrak{g}_0 & \rightarrow & \mathfrak{g} & \rightarrow & \mathfrak{so}(m) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \rightarrow & \mathfrak{k}_0 & \rightarrow & \mathfrak{k} & \rightarrow & \mathfrak{so}(m) \end{array}$$

where the vertical arrows are the inclusions. Below we prove that the bottom sequence extends to

$$0 \rightarrow \mathfrak{k}_0 \rightarrow \mathfrak{k} \rightarrow \mathfrak{so}(m) \rightarrow 0$$

if and only if \mathfrak{n} is type H . This is not the case for the top one: the condition that

$$0 \rightarrow \mathfrak{g}_0 \rightarrow \mathfrak{g} \rightarrow \mathfrak{so}(m) \rightarrow 0$$

is exact defines a class of fat algebras strictly larger than type H . We describe it in the next section for $m = 2$.

Proposition 2.2. *Let $\mathfrak{n} = \mathfrak{v} + \mathfrak{z}$ be an algebra of type H . There is a metric on \mathfrak{z} such that $\mathfrak{g}/\mathfrak{g}_0 \cong \mathfrak{so}(m)$.*

Proof. There is an inner product in \mathfrak{v} such that the $J_i = T_i$'s satisfy the Canonical Anticommutation Relations

$$J_w J_z + J_z J_w = -2 \langle z, w \rangle I.$$

For $\|z\| = 1$ let $r_z \in O(\mathfrak{z})$ be the reflection through the hyperplane orthogonal to z and $J_z \in SL(\mathfrak{v})$ be as above. Then

$$g_{(J_z, -r_z)} = \begin{pmatrix} J_z & 0 \\ 0 & -r_z \end{pmatrix} \in \text{Aut}(\mathfrak{n}).$$

Indeed,

$$\begin{aligned}
(w, [J_z u, J_z v]) &= (J_w J_z u, J_z v) = (-J_z J_w u - 2(z, w)u, J_z v) \\
&= -(J_z J_w u, J_z v) - 2(z, w)(u, J_z v) = (J_w u, J_z J_z v) + 2(z, w)(J_z u, v) \\
&= -(J_w u, v) + 2(z, w)(J_z u, v) = (J_{-w+2(z, w)z} u, v) \\
&= (-w + 2(z, w)z, [u, v]) = (-r_z(w), [u, v]) \\
&= (w, -r_z([u, v])),
\end{aligned}$$

so that

$$-r_z([u, v]) = [J_z u, J_z v].$$

The Lie group generated by the $-r_z$ has finite index in $O(\mathfrak{z})$. \square

Corollary 2.3. *Let \mathfrak{n} be a fat algebra with center of dimension m . Then*

$$\dim(K/K_0) \leq \dim(G/G_0) \leq m(m-1)/2$$

with equality achieved for any type H algebra of the same dimension with center of the same dimension.

Since $\text{Aut}(\mathfrak{n})/\text{Aut}_0(\mathfrak{n}) = (G/G_0) \times (\text{dilations})$, one obtains

Corollary 2.4. *Let \mathfrak{n} be a fat algebra with center of dimension m . Then*

$$\dim(\text{Aut}(\mathfrak{n})/\text{Aut}_0(\mathfrak{n})) \leq 1 + m(m-1)/2,$$

with equality achieved for any type H algebra of the same dimension and with center of the same dimension.

A converse for Corollary 2.3 is

Theorem 2.5. *If \mathfrak{n} is fat with center of dimension m and*

$$\dim(K/K_0) = m(m-1)/2$$

for some metric on \mathfrak{v} , then \mathfrak{n} is of type H .

Proof. The hypothesis implies that $\mathfrak{k}/\mathfrak{k}_0 = \mathfrak{g}/\mathfrak{g}_0 \cong \mathfrak{so}(m)$, so that K/K_0 acts transitively among the $|z| = 1$. For $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ in this group, $-T_{bz} = aT_z a^{-1}$, hence $T_{bz}^2 = aT_z^2 a^{-1}$. Since T_z is invertible, we can choose the metric such that $T_{z_0}^2 = -I$ for any given z_0 . Therefore $T_z^2 = -I$ for all $|z| = 1$, which implies the assertion. \square

Maximal dimension means there are isomorphisms

$$\mathrm{Lie}(K/K_0) = \mathrm{Lie}(G/G_0) \cong \mathfrak{so}(m).$$

Therefore the simply connected covers are isomorphic: $\mathrm{Spin}(m) \cong \widetilde{(G/G_0)}_e$. The induced homomorphism

$$\mathrm{Spin}(m) \rightarrow (G/G_0)_e$$

may or may not extend to a homomorphism

$$\mathrm{Pin}(m) \rightarrow G/G_0.$$

If it does extend, it may or may not be injective, in which case it is an isomorphism. Therefore, among the algebras for which $\dim(G/G_0)$ is maximal, those for which $\mathrm{Pin}(m) \cong G/G_0$ can be regarded as the most symmetric.

Theorem 2.6. *Suppose \mathfrak{n} is a 2-step graded algebra such that $\mathrm{Aut}(\mathfrak{n})$ contains a copy of $\mathrm{Pin}(m)$ inducing the standard action on \mathfrak{z} . Then \mathfrak{n} is type H.*

Proof. The assumption implies that there is a linear map $\mathfrak{z} \rightarrow \mathrm{End}(\mathfrak{v})$, denoted by $z \mapsto J_z$ such that $J_z^2 = -|z|^2 I$ for all z and

$$[J_z u, J_z v] = r_z([u, v])$$

for $u, v \in \mathfrak{v}$, $z \in \mathfrak{z}$, $|z| = 1$, where r_z is the reflection in \mathfrak{z} with respect of the line spanned by z . $\mathrm{Pin}(m)$ is the group generated by the J_z 's with $\|z\| = 1$, which acts linearly on \mathfrak{v} and is compact. Fix a metric on \mathfrak{v} invariant under it.

We get, as in the proof of Theorem 2.1, that if $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{Aut}(\mathfrak{n})$, then

$$T_{b^\dagger z} = a^\dagger T_z a.$$

In particular:

$$T_{r_x(z)} = J_x T_z J_x.$$

If $x = z$, we get $T_z = -J_z T_z J_z$, thus $T_z J_z = -J_z^{-1} T_z = J_z T_z$. If $x \perp z$, we get $T_z = J_x T_z J_x$, thus $T_z J_x = J_x^{-1} T_z = -J_x T_z$. It follows that T_z^2 commutes with J_z and with J_w , $w \perp z$.

Now, let $z \in \mathfrak{z}$ and $w \perp z$. Let $R_w(t)$ the $2t$ -rotation from z towards w . Then $R_w(t) = r_z r_{w(t)}$, with $w(t) = \cos(t)z + \sin(t)w$. It follows that

$$\begin{pmatrix} J_z J_{w(t)} & 0 \\ 0 & R_w(t) \end{pmatrix}$$

is an orthogonal automorphism and, therefore, satisfies

$$T_{R_w(t)z} = (J_z J_{w(t)})^\dagger T_z (J_z J_{w(t)}).$$

Since $(J_z J_{w(t)})^\natural = (J_z J_{w(t)})^{-1}$,

$$T_{R_w(t)z}^2 = (J_z J_{w(t)})^\natural T_z^2 (J_z J_{w(t)}) = J_{w(t)} J_z T_z^2 J_z J_{w(t)}.$$

Since T_z^2 commutes with J_z and J_w ,

$$(2.4) \quad T_{R_w(t)z}^2 = T_z^2 J_{w(t)} J_z J_z J_{w(t)} = -T_z^2 J_{w(t)} J_{w(t)}.$$

But $J_{w(t)}^2 = -I$, so that (2.4) implies that

$$T_{R_w(t)z}^2 = T_z^2.$$

For all $z' \in \mathfrak{z}$ we can choose $w \in \mathfrak{z}, t \in \mathbb{R}$ such that $R_w(t)z = z'$, so we get

$$T_{z'}^2 = T_z^2, \quad \text{for all } z' \in \mathfrak{z}, |z'| = 1.$$

The anti-symmetry of the bracket implies that T_z is skew-symmetric. Rescaling the scalar product on \mathfrak{v} we obtain that $T_z^2 = -I$, so $T_{z'}^2 = -I$ for all $z' \in \mathfrak{z}, |z'| = 1$. Therefore \mathfrak{n} is type H . □

3. THE CASE OF CENTER OF DIMENSION 2

In this section we compute the groups $G, G_0, G/G_0$ in the case $m = 2$. The various types are parametrized by pairs

$$(\mathbf{c}, \mathbf{r}) \in (\mathbb{U}^\ell / SL(2, \mathbb{R})) \times \mathbb{Z}_+^\ell$$

where \mathbb{U} is the upper-half plane and $2\ell = 2\sum r_j = \dim \mathfrak{n} - 2$. As a corollary we conclude that $\text{Aut}(\mathfrak{n})$ is maximal if and only if \mathfrak{n} is type H . These are complex Heisenberg algebras of various dimensions regarded as real Lie algebras.

First we recall the normal form for fat algebras with $m = 2$ deduced from [LT]. Given $c = a + bi \in \mathbb{C}$, let

$$Z(c) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

If $r \in \mathbb{Z}_+$, set

$$A(c, r) = \begin{pmatrix} Z(c) & & & \\ I_2 & Z(c) & & \\ & & \ddots & \\ & & & I_2 & Z(c) \end{pmatrix}$$

a $2r \times 2r$ -matrix. If $\mathbf{c} = (c_1, \dots, c_\ell) \in \mathbb{C}^\ell$ and $\mathbf{r} = (r_1, \dots, r_\ell) \in \mathbb{N}_+^\ell$, set

$$A(\mathbf{c}, \mathbf{r}) = \begin{pmatrix} A(c_1, r_1) & & & \\ & A(c_2, r_2) & & \\ & & \ddots & \\ & & & A(c_\ell, r_\ell) \end{pmatrix}$$

which is a $2s \times 2s$ matrix, $s = r_1 + \dots + r_\ell$.

Let now $\phi, \Psi_{(\mathbf{c}, \mathbf{r})}$ be the 2-forms on \mathbb{R}^{4s} whose matrices in the standard basis are

$$(3.1) \quad [\phi] = \begin{pmatrix} 0 & -I_{2s} \\ I_{2s} & 0 \end{pmatrix} \quad [\Psi_{(\mathbf{c}, \mathbf{r})}] = \begin{pmatrix} 0 & A(\mathbf{c}, \mathbf{r}) \\ -A^t(\mathbf{c}, \mathbf{r}) & 0 \end{pmatrix}.$$

Then

$$[u, v]_{(\mathbf{c}, \mathbf{r})} = (\phi(u, v), \Psi_{(\mathbf{c}, \mathbf{r})}(u, v)) = \langle u, [\phi]v \rangle e_1 + \langle u, [\Psi_{(\mathbf{c}, \mathbf{r})}]v \rangle e_2$$

is an \mathbb{R}^2 -valued antisymmetric 2-form on \mathbb{R}^{4s} . Let

$$\mathfrak{n}_{(\mathbf{c}, \mathbf{r})} = \mathbb{R}^{4s} \oplus \mathbb{R}^2$$

be the corresponding Lie algebra.

Define $M_{(\mathbf{c}, \mathbf{r})} \in \text{End}(\mathfrak{v})$ by

$$\phi(M_{(\mathbf{c}, \mathbf{r})}u, v) = \Psi_{(\mathbf{c}, \mathbf{r})}(u, v),$$

whose matrix is

$$[M_{(\mathbf{c}, \mathbf{r})}] = \begin{pmatrix} -A^t(\mathbf{c}, \mathbf{r}) & 0 \\ 0 & -A(\mathbf{c}, \mathbf{r}) \end{pmatrix}.$$

then we have

$$(3.2) \quad [u, v]_{(\mathbf{c}, \mathbf{r})} = \phi(u, v)e_1 + \phi(M_{(\mathbf{c}, \mathbf{r})}u, v)e_2, \text{ for } u, v \in \mathbb{R}^{4s}.$$

One can deduce [LT]

Proposition 3.1.

(a) Every fat algebra with center of dimension 2 is isomorphic to some $\mathfrak{n}_{(\mathbf{c}, \mathbf{r})}$ with $\mathbf{c} \in \mathbb{U}^\ell$.

(b) Two of these are isomorphic if and only if the \mathbf{r} 's coincide up to permutations and the \mathbf{c} 's differ by some Möbius transformation acting componentwise.

(c) $\mathfrak{n}_{(\mathbf{c}, \mathbf{r})}$ is of type H if and only if $\mathbf{c} = (c, \dots, c)$ and $\mathbf{r} = (1, \dots, 1)$

Let now

$$\mathfrak{n} = \mathfrak{n}_{(\mathbf{c}, \mathbf{r})}$$

be fat and let $G = G(\mathfrak{n})$, etc. We denote $\hat{\mathfrak{n}}$ the algebra obtained by replacing the matrices $A(c, r)$ by their semisimple parts and setting all $c_j = \sqrt{-1}$. The resulting $\hat{A}(c, r)$ consists of blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ along the diagonal and $\hat{\mathfrak{n}}$ is isomorphic to the H -type algebra $\mathfrak{n}_{((i, \dots, i), (1, \dots, 1))}$. The correspondence

$$\mathfrak{n} \mapsto \hat{\mathfrak{n}}$$

is functorial and seems extendable inductively to fat algebras of any dimension, although here we will maintain the assumption $m = 2$.

Lemma 3.2. $G_0(\mathfrak{n}) \subset G_0(\hat{\mathfrak{n}})$ and $\dim \text{Aut}(\mathfrak{n}) \leq \dim \text{Aut}(\hat{\mathfrak{n}})$.

Proof. Let $\phi, \psi, M_{(\mathbf{c}, \mathbf{r})} \in \text{End}(\mathfrak{v})$ be as above, so that

$$\phi(M_{(\mathbf{c}, \mathbf{r})}u, v) = \psi_{(\mathbf{c}, \mathbf{r})}(u, v).$$

By formula (3.2), $g \in G_0(\mathfrak{n}_{(\mathbf{c}, \mathbf{r})})$ if and only if

$$\phi(u, v) = \phi(gu, gv), \quad \phi(M_{(\mathbf{c}, \mathbf{r})}u, v) = \phi(M_{(\mathbf{c}, \mathbf{r})}gu, gv) = \phi(g^{-1}M_{(\mathbf{c}, \mathbf{r})}gu, v),$$

i.e., if and only if $g \in Sp(\phi)$ and commutes with $M_{(\mathbf{c}, \mathbf{r})}$. In particular it commutes with the semisimple part $\hat{M}_{(\mathbf{c}, \mathbf{r})}$. This is conjugate to a matrix having blocks $Z(c) = \begin{pmatrix} \Re(c) & \Im(c) \\ -\Im(c) & \Re(c) \end{pmatrix}$ for various $c \in \mathbb{C}$ along the diagonal, and zeros elsewhere. Every matrix commuting with such a matrix will surely commute with that having all $c = 1$. It follows that g also preserves $\phi(\hat{M}_{(\mathbf{c}, \mathbf{r})}u, v)$ and, therefore, it is an automorphism of $\hat{\mathfrak{n}}$ as well. Thus,

$$G_0(\mathfrak{n}) \subset G_0(\hat{\mathfrak{n}}).$$

From Corollary 2.3, $\dim(G(\mathfrak{n})/G_0(\mathfrak{n})) \leq \dim(G(\hat{\mathfrak{n}})/G_0(\hat{\mathfrak{n}}))$, and therefore

$$\dim G(\mathfrak{n}) = \dim(G(\mathfrak{n})/G_0(\mathfrak{n})) + \dim G_0(\mathfrak{n}) \leq \dim(G(\hat{\mathfrak{n}})/G_0(\hat{\mathfrak{n}})) + \dim G_0(\hat{\mathfrak{n}}) = \dim G(\hat{\mathfrak{n}}).$$

Formula (2.2) implies $\dim \text{Aut}(\mathfrak{n}) \leq \dim \text{Aut}(\hat{\mathfrak{n}})$, as claimed. \square

Next we will describe $\mathfrak{g}_0(\mathfrak{n}_{(\mathbf{c}, \mathbf{r})})$ for $c \in \mathbb{U}$ and $r \in \mathbb{N}_+$, i.e., the case when the matrices A consist of a single block. Since c is $SL(2, \mathbb{C})$ -conjugate to i , it is enough to take $c = i$. Define the 2×2 -matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and let $M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle)$ and $M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y} \rangle)$ denote the real vector spaces of $r \times r$ matrices with coefficients in the span of $\mathbf{1}, \mathbf{i}$ and \mathbf{x}, \mathbf{y} respectively. Then the vector space

$$\mathcal{R}(r) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, D \in M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle), B, C \in M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y} \rangle) \right\},$$

is actually a matrix algebra.

Note that

$$\mathbf{1}^t = \mathbf{1}, \quad \mathbf{i}^t = -\mathbf{i}, \quad \mathbf{x}^t = \mathbf{x}, \quad \mathbf{y}^t = \mathbf{y}.$$

Letting A^t denote the transpose of an \mathbb{R} -matrix and A^t, A^* the transpose and conjugate transpose of $\mathbb{R}[\mathbf{i}, \mathbf{x}, \mathbf{y}]$ -matrices, one obtains

$$A^t = A^*$$

for $A \in M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle)$ while

$$A^t = A^t$$

for $A \in M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y} \rangle)$.

With the notation

$$J_1 = [\phi] \quad J_2 = [\psi_{((i, \dots, i), (1, \dots, 1))}],$$

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) = \{X \in M_{4r}(\mathbb{R}) : J_1 X + X^t J_1 = 0, J_2 X + X^t J_2 = 0\}.$$

From [S] we know that

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) \cong \mathfrak{sp}(r, \mathbb{C})^{\mathbb{R}}$$

Changing basis,

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) = \{X \in \mathcal{R}(r) : J_1 X + X^t J_1 = 0, J_2 X + X^t J_2 = 0\}$$

where

$$J_1 = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & \mathbf{i}I_r \\ \mathbf{i}I_r & 0 \end{pmatrix}.$$

This gives an alternative description of this algebra:

$$\mathfrak{g}_0(\hat{\mathfrak{n}}) = \left\{ \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} : A \in M_r(\mathbb{R}\langle \mathbf{1}, \mathbf{i} \rangle), B, C \in M_r(\mathbb{R}\langle \mathbf{x}, \mathbf{y} \rangle), B^t = B, C^t = C \right\}$$

We now restrict our attention to matrices $\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$ in $\mathfrak{g}_0(\hat{\mathfrak{n}})$ where A, B, C have the respective forms

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{pmatrix} \quad \begin{pmatrix} b_1 & \cdots & b_{r-1} & b_r \\ \vdots & \ddots & \ddots & 0 \\ b_{r-1} & \ddots & \ddots & \vdots \\ b_r & 0 & \cdots & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & \cdots & 0 & c_1 \\ \vdots & \ddots & \ddots & c_2 \\ 0 & \ddots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_r \end{pmatrix}$$

with coefficients in $M_2(\mathbb{R})$. Let $\mathbf{A}_k = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$ having $a_k = \mathbf{1}$ and zero otherwise and \mathbf{A}'_k the matrix of the same form but with $a_k = \mathbf{i}$ and zeros elsewhere. Similarly, let \mathbf{B}_k (resp. \mathbf{C}_k) the matrix $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ (resp., $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$) with b_k (resp. c_k) equal to \mathbf{x} and zeros elsewhere, and \mathbf{B}'_k (resp. \mathbf{C}'_k) with b_k (resp. c_k) equal to \mathbf{y} and zeros elsewhere.

Theorem 3.3. *Let $\mathfrak{n} = \mathfrak{n}_{(c,r)}$, $(c, r) \in \mathbb{U} \times \mathbb{N}$, and regard $\mathfrak{g}_0(\mathfrak{n})$ as a subalgebra of $\mathfrak{gl}(\mathfrak{v})$. Then,*

- (1) $\mathfrak{g}_0(\mathfrak{n})$ is the \mathbb{R} -span of $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$ for $1 \leq i \leq r$.
- (2) The semisimple part of $\mathfrak{g}_0(\mathfrak{n})$ is the span of $\mathbf{A}_1, \mathbf{A}'_1, \mathbf{B}_1, \mathbf{B}'_1, \mathbf{C}_1, \mathbf{C}'_1$.
- (3) The solvable radical is the span of $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$ with $1 < i \leq r$.

In particular, the \mathbb{R} -dimension of $\mathfrak{g}_0(\mathfrak{n})$ is equal to $6r$ and the semisimple part of $\mathfrak{g}_0(\mathfrak{n})$ is isomorphic to $\mathfrak{sp}(1, \mathbb{C})$.

Proof. It is enough to consider the case $\mathfrak{n} = \mathfrak{n}_{(i,r)}$. Let $T_2 = [\Psi_{(i,r)}]$ and write $T_2 = J_2 + N_2$ where

$$N_2 = \begin{pmatrix} 0 & N \\ -N^t & 0 \end{pmatrix}, \text{ with } N = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 1 & \ddots & 0 & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & 1 & 0 \end{pmatrix}.$$

From Lemma 3.2, $\mathfrak{g}_0(\mathfrak{n}) = \{X \in \mathfrak{g}_0(\hat{\mathfrak{n}}) : T_2 X + X^t T_2 = 0\}$. As $\mathfrak{g}_0(\mathfrak{n}) \subset \mathfrak{g}_0(\hat{\mathfrak{n}})$ one obtains

$$\mathfrak{g}_0(\mathfrak{n}) = \{X \in \mathfrak{g}_0(\hat{\mathfrak{n}}) : N_2 X + X^t N_2 = 0\}.$$

The conditions on $\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \in \mathfrak{g}_0(\mathfrak{n})$ are, explicitly,

$$(3.3) \quad 0 = NC - C^t N^t = NC - (NC)^t$$

$$(3.4) \quad 0 = N^t A - A N^t$$

$$(3.5) \quad 0 = N^t B - B^t N = N^t B - (N^t B)^t.$$

For the first equation, note that NC symmetric if and only if $c_{i,j+1} = c_{j,i+1}$ and $c_{1,j} = 0$ for $i, j < n$. Since C is symmetric, $c_{i,j+1} = c_{j,i+1} = c_{i+1,j}$ and $c_{1,j} = 0$ for $i, j < n$. We conclude:

$$\text{If } i + j = k \leq r, c_{i,j} = c_{i,k-i} = c_{i-1,k-i+1} = c_{i-2,k-i+2} \cdots = c_{1,k-1} = 0$$

$$\text{If } i + j = k > r, c_{i,j} = c_{i,k-i} = c_{i+1,k-i-1} = c_{i+2,k-i-2} \cdots = c_{r,k-i-i-r} = c_{r,k-r}$$

Thus, the strict upper antidiagonals are zero and each lower antidiagonal have all its elements equal.

For the second equation, note that N^t and A commute. This is equivalent to $c_{i,j} = c_{t,s}$ when $j - i = s - t$ and $c_{i,1} = 0$ for $i > 1$. The first condition implies that each diagonal have all its elements equal, while the second implies that the strict lower diagonals are zero.

Equation (3.5) is analogous to equation (3.3): the condition $N^t B$ symmetric is equivalent to each antidiagonal have all its elements equal and that the strict lower antidiagonals are 0.

From all this we conclude that the span of $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$ with $1 \leq i \leq r$ is $\mathfrak{g}_0(\mathfrak{n})$ and (1) follows.

(2) and (3) follow from (1) and the explicit presentation of the matrices $\mathbf{A}_i, \mathbf{A}'_i, \mathbf{B}_i, \mathbf{B}'_i, \mathbf{C}_i, \mathbf{C}'_i$.

□

Corollary 3.4. *(of the proof) Let \mathfrak{n} be fat. Then $\dim(\mathfrak{g}_0(\mathfrak{n}))$ is maximal if and only if \mathfrak{n} is of H -type.*

Proof. Let $(\mathbf{c}, \mathbf{r}) = ((c_1, \dots, c_l), (r_1, \dots, r_l))$ be such that $\mathfrak{n} = \mathfrak{n}_{(\mathbf{c}, \mathbf{r})}$. We know that $\mathfrak{g}_0(\mathfrak{n}) \subset \mathfrak{g}_0(\hat{\mathfrak{n}})$. If $c_i \neq c_j$ for some i, j , then there is not intertwining operator between the blocks corresponding to these invariants, so $\mathfrak{g}_0(\mathfrak{n}) \neq \mathfrak{g}_0(\hat{\mathfrak{n}})$.

When $c_1 = c_2 = \dots = c_l$ we can consider $c_j = i$ for all j . Let $r = \sum r_i$. In this case if $\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \in \mathfrak{g}_0(\mathfrak{n})$ must satisfy the equations (3.3), (3.4), (3.5) but with N such that coefficients $n_{j+1,j}$ are 0 or $\mathbf{1}$. Suppose now that $\mathfrak{g}_0(\mathfrak{n})$ is not of H -type, then some $n_{j+1,j}$ is equal to $\mathbf{1}$. We assume that $n_{21} = \mathbf{1}$ and let $A \in M_r(\mathbb{R}[\mathbf{i}])$ such that $a_{12} = \mathbf{1}$ and 0 otherwise, then

$$X = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}$$

belongs to $\mathfrak{g}_0(\hat{\mathfrak{n}})$ but is not in $\mathfrak{g}_0(\mathfrak{n})$. □

It can be shown in general that the semisimple part of $\mathfrak{g}_0(\mathfrak{n})$ is isomorphic to $\oplus_i \mathfrak{sp}(m_i, \mathbb{C})$, where m_i is the multiplicity of the pair (c_i, r_i) in (\mathbf{c}, \mathbf{r}) .

In the case $m = 2$, $\mathfrak{g}/\mathfrak{g}_0$ is either 0 or isomorphic to $\mathfrak{so}(2)$.

Theorem 3.5. $\mathfrak{g}(\mathfrak{n})/\mathfrak{g}_0(\mathfrak{n}) \cong \mathfrak{so}(2)$ if $c_1 = \dots = c_\ell$, and 0 otherwise.

Proof. $\mathfrak{g}/\mathfrak{g}_0$ is a compact subalgebra of $\mathfrak{gl}(2)$, hence of the form $g\mathfrak{so}(2)g^{-1}$ for some $g \in SL(2, \mathbb{R})$ and it is nonzero if and only if there exists $X \in \mathfrak{sl}(v)$ such that, in the notation of the proof of Theorem 3.3,

$$\begin{pmatrix} X & 0 \\ 0 & g\mathbf{i}g^{-1} \end{pmatrix}$$

is a derivation of \mathfrak{n} . For $g = \mathbf{1}$, if T_1, T_2 correspond to the standard basis of \mathfrak{z} , the equations for X become

$$(a) \quad T_1 X + X^t T_1 = T_2, \quad (b) \quad T_2 X + X^t T_2 = -T_1$$

In normal form, and for a single block $A_{(i,r)}$,

$$T_1 = J_1 = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & \mathbf{i}I_r + N \\ \mathbf{i}I_r - N^t & 0 \end{pmatrix}.$$

We decompose

$$T_2 = J_2 + N_2, \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & \mathbf{i}I_r \\ \mathbf{i}I_r & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & N \\ -N^t & 0 \end{pmatrix}$$

and regard J_1, J_2, T_1, T_2, N_2 as matrices with coefficients in $M_2(\mathbb{R})$. Note that J_1, J_2 correspond to \hat{n} , of type H . Let

$$Y_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & s & 0 \\ 0 & 2\mathbf{i} & 0 & 0 & 0 & & s & 0 \\ 0 & \mathbf{1} & 4\mathbf{i} & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 2\mathbf{1} & 6\mathbf{i} & 0 & 0 & s & 0 \\ 0 & 0 & 0 & 3\mathbf{1} & 8\mathbf{i} & & s & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \vdots & \vdots & 0 & (n-2)\mathbf{1} & 2(n-1)\mathbf{i} \end{pmatrix}.$$

A straightforward calculation shows that

$$X_0 = \begin{pmatrix} -Y_0^\dagger & 0 \\ 0 & -Y_0^\dagger + \mathbf{i}I_r + N \end{pmatrix}$$

is a solution of (a), (b). We conclude that

$$\begin{pmatrix} X_0 & 0 \\ 0 & \mathbf{i} \end{pmatrix}$$

is a derivation of $\mathfrak{n}_{(i,r)}$, which lies in $\mathfrak{g}(\mathfrak{n}_{(i,r)})$ but not in $\mathfrak{g}_0(\mathfrak{n}_{(i,r)})$.

For any $c \in \mathbb{U}$, $\mathfrak{n}_{(c,r)} \cong \mathfrak{n}_{(i,r)}$, hence they have the same $\mathfrak{g}/\mathfrak{g}_0$ up to isomorphisms. In fact, for any $g \in Sl(2, \mathbb{R})$, the algebra $\mathfrak{n}_{(g \cdot i, r)}$ has a derivation of the form

$$\begin{pmatrix} X & 0 \\ 0 & g\mathbf{i}g^{-1} \end{pmatrix}.$$

For a fixed g , these X are unique modulo \mathfrak{g}_0 and come in normal form. Clearly, c determines the 2×2 matrix $g\mathbf{i}g^{-1}$ and the complex number $g \cdot i$.

In the case of an arbitrary fat $\mathfrak{n}_{(\mathbf{c}, \mathbf{r})}$, each block (c_k, r_k) determines a corresponding X_k such that

$$\begin{pmatrix} X_k & 0 \\ 0 & g_k\mathbf{i}g_k^{-1} \end{pmatrix}$$

is a derivation of $\mathfrak{n}_{(c_k, r_k)}$. If $\mathfrak{n}_{(\mathbf{c}, \mathbf{r})}$ has a derivation in \mathfrak{g} that is not in \mathfrak{g}_0 , then it must have one which is combination of such, acting on \mathfrak{v} as $X_1 + X_2 + \dots$. This forces all the $g_k\mathbf{i}g_k^{-1}$ to be the same and all the c_i to be the same. The reciprocal is clear. \square

In particular, all algebras $\mathfrak{n}_{(\mathbf{c}, \mathbf{r})}$ with $c_1 = \dots = c_\ell$ and $r_i > 1$ maximize the dimension of $\mathfrak{g}/\mathfrak{g}_0$, but they are not type H .

Lauret had pointed out to us that there were non-type H algebras such that $\mathfrak{g}(\mathfrak{n})/\mathfrak{g}_0(\mathfrak{n}) \neq 0$. Independently, Oscari proved that this holds whenever the c_i 's all agree.

4. FAT DISTRIBUTIONS

Let D be a smooth vector distribution on a smooth manifold M , i.e., a subbundle of the tangent bundle $T(M)$. Its nilpotentization, or symbol, is the bundle on M with fiber

$$N^D(M)_p = \bigoplus_j D_p^{(j)} / D_p^{(j-1)}$$

where $D_p^{(1)} = D_p$ and $D_p^{(j+1)} = D_p^{(j)} + [\Gamma(D), \Gamma(D^j)]_p$. The Lie bracket in $\Gamma(T(M))$ induces a graded nilpotent Lie algebra structure on each fiber of $N^D(M)$. If $D^{(j)} = T(M)$ for some j , D is called completely non-integrable. If $D^{(2)} = T(M)$, the nilpotentization is 2-step, which in the notation of the previous section, is

$$\mathfrak{n}_p = N_D(M)_p = D_p \oplus \frac{D_p + [\Gamma(D), \Gamma(D)]_p}{D_p} = \mathfrak{v}_p + \mathfrak{z}_p,$$

It is also easy to see that D is fat in the sense of Weinstein [M] if and only if $\mathfrak{n}_p = \mathfrak{v} + \mathfrak{z}$ is non-singular, i.e., fat in the sense defined in the section 1.

A subriemannian metric g defined on D determines a metric on \mathfrak{v} . On \mathfrak{z} we put a metric σ invariant under G . Let $\{\phi_1, \dots, \phi_m; \psi_1, \dots, \psi_n\}$ be a coframe on M such that

$$D = \cap \ker \phi_i,$$

with $\{\phi_1, \dots, \phi_m\}$ and $\{\psi_1, \dots, \psi_n\}$ orthonormal with respect to $g + \sigma$. Define $T_z \in \text{End}(D)$ as before, by

$$\sigma(z, [u, v]) = g(T_z u, v).$$

Then D is fat if and only if T_z is invertible for all non-zero $z \in \mathfrak{z}$. The structure equations for the coframe can be written

$$d\phi_k \equiv \sum_i (T_k \psi_i) \wedge \psi_i \quad \text{mod}(\phi_\ell)$$

with the T_k 's having the property that any non-zero linear combination of them is invertible. This is deduced from the fact that if $u, v \in \mathfrak{v}$, then $d\phi[u, v] = -\phi([u, v])$, since $u(\phi(v)) = u(0) = 0$. The $d\psi$'s are essentially arbitrary.

Let now M be a the simply connected Lie group with a fat Lie algebra \mathfrak{n} , D the left-invariant distribution on M such that $D_e = \mathfrak{v}$. For a left-invariant coframe, the structure equations take the form

$$d\phi_k = \sum_i (J_k \psi_i) \wedge \psi_i, \quad d\psi_i = 0$$

where J_1, \dots, J_m are anticommuting complex structures on D .

The results from the previous sections lead to consider fat distributions satisfying

$$(4.1) \quad d\phi_k = \sum_i (J_k \psi_i) \wedge \psi_i \quad \text{mod}(\phi_\ell)$$

where the J_k are sections of $\text{End}(T(M)^*)$ satisfying the Canonical Commutation Relations

$$J_i J_j + J_j J_i = -2\delta_{ij}.$$

The Equivalence Problem for these systems has been discussed for distributions with growth vector $(2n, 2n+1), (4n, 4n+3)$ and $(8, 15)$. In these cases \mathfrak{n} is parabolic, i.e., isomorphic to the Iwasawa subalgebra of a real semisimple Lie algebra \mathfrak{g} of real rank one. The Tanaka [T] subriemannian prolongation of such algebra is \mathfrak{g} , while in the non-parabolic case is just

$$\mathfrak{n} + \mathfrak{k}(\mathfrak{n}) + \mathfrak{a}(\mathfrak{n})$$

where $\mathfrak{a}(\mathfrak{n})$ the 1-dimensional Lie algebra of dilations [Su]. In this case, Tanaka's theorem implies that, in the notation of [Z], the first pseudo G-structure P^0 already carries a canonical frame.

As this paper was being written, E. van Erp pointed out to us his article [Er], where fat distributions are called polycontact and those satisfying (4.1) arise by imposing a compatible conformal structure.

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FACULTAD DE MATEMÁTICA, ASTRONOMÍA Y FÍSICA, UNIVERSIDAD NACIONAL DE CÓRDOBA. CIEM – CONICET. (5000) CIUDAD UNIVERSITARIA, CÓRDOBA, ARGENTINA
E-mail address: (kaplan, tirabo)@famaf.unc.edu.ar